BRUHAT INTERVALS AND POLYHEDRAL CONES

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Abstract. Lecture notes by Matthew Dyer for lectures at the workshop "Coxeter groups and convex geometry."

1. Root labelled Bruhat graph

1.1. **Bruhat order.** Let (W, S) be a Coxeter group. Bruhat order is the partial order defined by the following proposition; see [3] and [13] for more details.

Proposition. *There is a partial order $\leq$ on W with the following properties. Let $w \in W$ and $w = s_1 \dots s_n$, $n = l_0(w)$, $s_i \in S$ be a reduced expression for w. Then $v \leq w$ if and only if there exist $m \leq n$ and $1 \leq i_1 < \dots < i_m \leq n$ such that $v = s_{i_1} \cdots s_{i_m}$. This statement is also true with an extra condition $m = l(v)$*

1.2. **Root-labelled Bruhat graph.** Fix a based root system (Φ, Π) with respect to $(V, \langle -, - \rangle)$, with associated Coxeter system (W, S). For some purposes, more general notions of root system including those of algebraic groups, Kac-Moody Lie algebras etc would be more natural. Let l_0 denote the standard length function of (W, S). We fix a length function $l \colon W \to \mathbb{Z}$ by $l = \pm l_0$ (there are many other suitable l; we shall not discuss them though they are required for proofs of some results even for l_0).

1.3. **Bruhat graph.** Bruhat order may also be defined in the following equivalent way.

Definition. (a) Define a directed graph $\Omega = \Omega_{(W,S,l)}$ with vertex set W and edge set $E = \{ (x, s_\alpha x) \mid x \in W, l(s_\alpha x) > l(x) \}$. Give $\Omega_{(W,S,l)}$ an edge labelling by Φ_+ such that the edge (x, y) receives label $\alpha \in \Phi_+$ if $y = s_\alpha x$, denoted $x \xrightarrow{\alpha} y$.

(b) The order $\leq_{(W,S,l)}$, or $\leq_l$, or $\leq$, is the partial order on W such that $x \leq y$ if there is a path $x = x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} x_n = y$ from x to y in Ω.

If $l = l_0$, these are called the *(root-labelled) Bruhat graph* and *Bruhat order*; for $l = -l_0$, one has the *reverse Bruhat graph* and *reverse Bruhat order*. We write (in any poset) $x \lessdot y$ if y covers x. In the above orders, this implies $l(y) = l(x) + 1$.

Exercise. Describe the graph and its labelling for some (small) dihedral groups, intervals of length ≤ 3, interval $[e, srts]$ in universal (W, S) (no braid relations).

Remarks. (1) The Bruhat order has, in important special cases, geometric and representation-theoretic interpretations in terms of inclusions of Schubert varieties and embeddings of Verma modules.

1.4. **Functoriality of Bruhat graph.** Let W' be a reflection subgroup with canonical based root system $(\Phi_{W'}, \Pi_{W'})$, canonical simple reflections S'.

Theorem. *Fix $l = l_0$ (resp., $-l_0$) and write $\leq$, Ω for $\leq_{(W,S,l)}$, $\Omega_{(W,S,l)}$. Let $l' := \pm l_{0,W'}$ be the corresponding length function for W'. Let $w \in W'$.*

(a) *$W'w$ has a unique minimal (resp., maximal) element x in $\leq$.*
(b) *The map $y \mapsto yx$ induces an isomorphism of edge-labelled directed graphs from $\Omega_{(W',S',l')}$ to the full subgraph of $\Omega_{(W,S,l)}$ on vertex set $W'w = W'x$.*

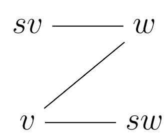
This has many application to combinatorics of Bruhat order when applied with W' ranging over dihedral reflection subgroups.

Exercise. Check the theorem for (W, S) of type B_2, say $W = \langle\, s, t \mid s^2 = t^2 = (st)^4 = 1 \,\rangle$ and $W' = \langle\, t, sts \,\rangle$ which is a reflection subgroup of type $A_1 \times A_1$. .

1.5. Z**-property (or lifting property) of Bruhat order.** The following property plays an important role in inductive arguments involving Bruhat intervals.

Theorem. *Let $v \leq w$ and $s \in S$ with $v < sv$ and $sw < w$:*

(1.5.1)



Then the following conditions are equivalent:

(a) $v \leq w$
(b) $sv \leq w$
(c) $v \leq sw$

Exercise. Prove it, or see [3].

1.6. For definitions of abstract simplicial complexes and their geometric realization, order complex $\Sigma(P)$ of a finite poset P, regular CW complexes and their face posets and basic properties of the face poset of a regular CW-complex underlying a ball, see [4, Appendix 3.7]. We use a non-standard convention of including an empty cell, so the face poset has a minimum element.

1.7. The topology of the order complex of Bruhat intervals is as follows.

Theorem. *Let $v \leq w$ and $n := l(w) - l(v)$. Then the closed interval $[v, w]$ is the face poset of a regular CW-complex underlying a ball of dimension $l(w) - l(v) - 1$. In particular, any maximal chain $v = v_0 < \ldots < v_n = w$ satisfies $n = l(w) - l(v)$, and if $n \geq 2$, then the order complex $\Sigma(v, w)$ of the open interval (v, w) is a combinatorial sphere of dimension $n - 2$.*

Proof. The main point is to establish CL-shellability (as in Björner and Wachs [5]) or EL-shellability [9], or give a recursive construction of a suitable regular CW complex using the Z-property as in [8] or [7]. For a quick introduction to shellability techniques, see [4, Appendix 3.7] □

Exercise. Why does the result hold for intervals of length 2?

2. Vector-labeled face lattice of a polyhedral cone

2.1. We assume familiarity with the definitions and basic properties of polyhedral cones, their faces and face lattices and the duality theorem for polyhedral cones.

General references for this material are are [16], [1]. We say a cone C is salient if $C\cap -C=\{0\}$ i.e. $0\in C$ but C contains no subspace of positive dimension (sometimes this condition is called pointedness of C but the terminology is not uniform).

2.2. Let C be a polyhedral cone in a finite-dimensional real vector space V. Let $\langle -,-\rangle$ be an inner product on V and use it to identify the dual cone $C^\vee$ of C with a subset of V: $C^\vee=\{\, v\in V \mid \langle v, C\rangle \subseteq \mathbb{R}_{\geq 0}\,\}$. The face of $C^\vee$ dual to a face x of C is $C^\vee\cap x^\perp$. Let F denote the face lattice of C, with partial order $\leq$ by inclusion, and define $l\colon F\to\mathbb{Z}$ by $l(x):=\dim(x):=\dim\mathbb{R}x$ for $x\in F$. Define a directed graph Ω_C with vertex set F and edges (x,y) for $x,y\in F$ with $x\leq y$ and $l(y)=l(x)+1$. Label the edge (x,y) of Ω by the inner unit normal to x in the linear span of y: $x\xrightarrow{\alpha} y$ if $\alpha\in\mathbb{R}y$, $\langle x,\alpha\rangle=0$, $\langle \alpha, y\rangle\subseteq\mathbb{R}_{\geq 0}$ and $\langle\alpha,\alpha\rangle=1$.

Remarks. (a) For any path $x=x_0\xrightarrow{\alpha_1} x_1\xrightarrow{\alpha_2}\dots\xrightarrow{\alpha_n} x_n=y$ in Ω_C, $\{\alpha_1,\dots,\alpha_n\}$ is an orthonormal basis of $\mathbb{R}y\cap x^\perp$.
(b) If one identifies the face lattice of $C^\vee$ with the opposite poset F^{op} by $x\mapsto C^\vee\cap x^\perp$, then $\Omega_{C^\vee}$ identifies naturally with the opposite labelled graph Ω^{op} (defined in the natural way).

Exercise. Describe the labelled face lattice for a two-dimensional cone.

2.3. It is well known that the topology of order complexes of polyhedral cones is as follows.

Theorem. *Let $v\leq w$ and $n:=l(w)-l(v)$. Then the closed interval $[v,w]$ is the face poset of a regular CW-complex underlying a ball of dimension $l(w)-l(v)-1$. In particular, any maximal chain $v=v_0<\ldots<v_n=w$ satisfies $n=l(w)-l(v)$, and if $n\geq 2$, then the order complex $\Sigma(v,w)$ of the open interval (v,w) is a combinatorial sphere of dimension $n-2$.*

Proof. Follows easily from the fact a convex polytope is homeomorphic to ball of the same dimension and its boundary is the union of its lower dimensional faces. One can also give a CL-shelling proof (Bruggeser-Mani, discussed in [4, Appendix 3.7]). A proof by CL-shelling techniques can be given of Theorems 1.5 and 2.3 using common properties of the labelings of Bruhat intervals and face lattices (see [11]). □

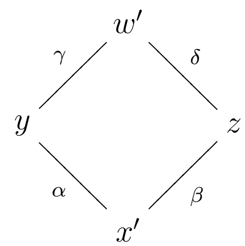
2.4. Some basic common properties of labellings of Bruhat interval and polyhedral cone are as follows (see [11]).

Proposition. *Consider an interval $[x,w]$ in $\leq_l$ or face lattice of polyhedral cone. Write $\lessdot$ for the covering relation.*

(a) *$\mathbb{R}_{\geq 0}\{\, \alpha \mid x \xrightarrow{\alpha} y \leq w \,\}$ is a salient polyhedral cone $C^{\uparrow}_{x,w}$ with $\{\, \alpha \mid x \xrightarrow{\alpha} y \leq z, x \lessdot y \,\}$ as set of representatives of its extreme rays. (Note in the cone case, $x \xrightarrow{\alpha} y$ implies $x \lessdot y$).*

(b) *Dually, $\mathbb{R}_{\geq 0}\{\, \alpha \mid x \leq y \xrightarrow{\alpha} w \,\}$ is a salient polyhedral cone $C^{\downarrow}_{x,w}$with $\{\, \alpha \mid x \leq y \xrightarrow{\alpha} z, y \lessdot w \,\}$ as set of representatives of its extreme rays*

(c) *Suppose $[x', w']$ is a length two subinterval of $[x, w]$, so its Hasse diagram and labeling are as follows*



Then $\{\alpha, \gamma\}$ and $\{\beta, d\}$ are linearly independent and $\begin{pmatrix} \gamma \\ \alpha \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \delta \\ \beta \end{pmatrix}$ where $a, b, c, d \in \mathbb{R}$ satisfy $b > 0, c > 0$, $ad - bc = -1$. In the crystallographic Coxeter group case (i.e. where $\langle\, \alpha_0, \alpha_1^{\vee} \,\rangle \in \mathbb{Z}$ for all $\alpha_0, \alpha_1 \in \Pi$), one has $a, b, c, d \in \mathbb{Z}$.

For cones, the proof is easy. The proof of (a)-(b) in the Coxeter group case uses analogues of Bruhat order and reverse Bruhat order defined in the same way as in 1.3 but for more general functions l than $l = \pm l_0$. It would be desirable to have a simpler proof. The proof of (c) reduces to the case of two dimensional cones, and intervals in dihedral groups (using functoriality of the Bruhat graph for the latter) where it can be checked by simple calculation.

2.5. Labelings of face lattices of polyhedral cones similar to those above but with respect to a more general bilinear form arise naturally in relation to Bruhat intervals, but are not well understood in general. Here are some natural questions about such labelings.

Question. (a) Suppose that $\langle\, -, - \,\rangle \colon V \times V \to \mathbb{R}$ is assumed to just be a non-degenerate bilinear form (not necessarily symmetric). What extra properties are required for labelling as above to be defined and to have similar properties to those for case of positive semidefinite inner product. For example, do they hold for a "generic" bilinear form?

(b) In the case of a Bruhat interval $[x, y]$, the polyhedral cone of non-negative linear combination of labels of edges of the Hasse diagram contains no line (since it is contained in the cone spanned by Φ_+). Is this true for Ω_C for any (or some) labelling induced by an inner product (or bilinear form as in (a)).

(c) If (b) has an affirmative answer, can a proof of dual EL-shellability of face posets of polyhedral cones be obtained by imitating the proof for Bruhat intervals? (I do not even know if such face lattices are EL-shellable, in general).

Remarks. Similar edge-labelled graphs satisfying certain conditions arise as moment graphs (see Braden and MacPherson [6]). In general, under mild conditions on such a graph, there is an associated representation category, which for natural graphs is often closely related to associated natural geometric or representation theoretic categories. The categories can be constructed from the labelled poset as in [6] or as an instance of a more general construction in [12]. One of the main goals of this ideas described here is is to relate such graphs from Bruhat intervals and polyhedral cones in order to construct functors between associated representation categories.

2.6. It can be shown that an interval $[x, y]$ in $\leq_l$ which has no length three subinterval with two coatoms is isomorphic to the face lattice of a polyhedral cone. The proof is by a lengthy and technical induction using the Z-property. A sufficiently nice answer to the following question could yield a simpler proof of a more general fact.

Question. Suppose given a poset $[x, w]$ which is the face poset of a regular CW-complex underlying a ball and with a labeling of edges of the Hasse diagram of $[x, w]$ by elements of a real vector space V, satisfying properties like those in 2.4(a)–(c). Suppose that the labels of edges in some (hence any) maximal chain from x to w are linearly independent. What (minimal) conditions ensure there is a poset isomorphism from $[x, w]$ to the face lattice of $C^{\uparrow}_{x,w}$ taking $y \in [x, w]$ to $C^{\uparrow}_{x,y}$? One would also want the conditions to imply that $C^{\uparrow}_{x,y}$ is "naturally" dual to $C^{\downarrow}_{x,y}$, and that the condition be inherited by closed subintervals etc.

3. Deodhar's conjecture

3.1. For a polyhedral cone C and $x \leq y \leq z$ in F, one has

$$|\{\, w \mid x \leq w \to y \text{ or } y \to w \leq z \,\}| \geq l(z) - l(x). \tag{3.1.1}$$

The proof of this reduces by duality to the case $y = x$, when it is equivalent to the statement that a $d = l(z) - l(y)$-dimensional polyhedral cone has at least d extreme rays. Equivalently, by homogenization, it is enough to show a $(d-1)$-dimensional polytope has at least d extreme points, which is trivial.

Theorem (Deodhar's conjecture)**.** *For $x \leq y \leq z$ in W,* (3.1.1) *holds.*

3.2. Some (informal) motivation. For a semisimple complex algebraic group G (e.g. SL_n) and Borel subgroup G (upper triangular matrices), the flag variety G/B has a natural structure of complete, complex algebraic variety. In the analytic topology, it is a compact, complex manifold. Let $T \subseteq B$ be a maximal torus (diagonal subgroup) and $W = N(T)/T$ be the Weyl group, which is a finite (crystallographic) Coxeter group. (It is S_n for SL_n). The Bruhat decomposition $G = \dot{\cup}_{w \in W} BwB$ of G into double cosets of B gives rise to a (CW) cell decomposition of G/B with cells $BwB/B \cong \mathbb{C}^{l(w)}$ where $l = l_0$. The closure $X(w)$ of BwB/B is an (in general singular) algebraic variety called a Schubert variety. One has $X(v) \subseteq X(w)$ if and only if $v \leq w$ (Bruhat order). The CW-structure given by Schubert varieties provide a natural basis for the cohomology ring $H^{\bullet}(G/B, \mathbb{R})$ indexed by W. Another description of this cohomology as ring was given by Borel; it is isomorphic to R/I

where R is the symmetric algebra on V (i.e. ring of polynomials on a basis V) and I is the ideal of R generated by W invariant polynomials of positive degree. The relationship between these pictures was described algebraically by Bernstein-Gelfand-Gelfand (see [2]), and more generally by Kostant and Kumar [14] in terms of what they call the nil Hecke ring of W. The first non-trivial case of Deodhar's conjecture (type A) was proved by Deodhar using the fact the dimension of the tangent space of an irreducible variety at each point is not less than the dimension of the variety (equality holds at smooth points, so equality in Deodhars conjecture provides a necessary condition for smoothness at certain points of type A Schubert varieties).

3.3. We return to the case of general W. Let $R := \mathrm{Sym}(V)$ denote the symmetric algebra of V, $\mathbb{N}$-graded so $R_0 = \mathbb{R}$ and $R_2 = V$. Hence R_{2n} is the set of homogeneous polynomials of degree n in the usual sense on a basis of V, and $R_{2n+1} = 0$. (The even grading will play no role in these talks, but for instance, makes R trivially "graded commutative" which is appropriate for its relation to cohomology of Schubert varieties etc). Let Q be the quotient field of R. Form the subring Q_W of functions $Q \to Q$ generated by actions of all elements of w on Q and multiplications by elements of Q. Let δ_w denote map $q \mapsto w(q)\colon Q \to Q$. Then Q_W has the elements δ_w for $w \in W$ as right (or left) Q-basis. The multiplication is determined by $(\delta_v q_v)(\delta_w q_w) = \delta_{vw} w^{-1}(q_v) q_w$. There is a ring anti-automorphism $h \mapsto \bar{h}$ of Q_W given by $\delta_w q_w \mapsto q_w \delta_{w^{-1}}$.

The nil Hecke ring H is the subring of Q_W consisting of functions in Q_W which map $R \to R$.

Exercise. For $\alpha \in \Phi$, $x_\alpha := \frac{1}{\alpha}(\delta_{s_\alpha} - \delta_e)$ is in H. One has $x_\alpha^2 = 0$ and for $\chi \in V$, $\chi x_\alpha = x_\alpha s_\alpha(\chi) - \langle \chi, \alpha^\vee \rangle \delta_e$.

The elements x_α for simple α can be shown to satisfy the braid relations, giving rise to elements x_w for $w \in W$ with $x_w = x_{\alpha_1} \cdots x_{\alpha_n}$ whenever $w = s_{\alpha_1} \cdots s_{\alpha_n}$ is reduced, $\alpha_i \in \Pi$. These elements from a (left or right) R-basis of the nil Hecke ring.

Exercise. Check the x_α satisfy the braid relations in type A_2 by direct computation.

3.4. More motivation. If W is finite, the x_α (and hence x_w) preserve the ideal I generated by invariant polynomials of positive degree and hence pass to operators on the coinvariant algebra R/I. For finite Weyl groups, these operators were defined by Demazure and Bernstein-Gelfand-Gelfand for study of the cohomology ring of flag varieties. For instance, the basis of cohomology from the Schubert cells can be constructed from the (unique) top dimensional class using them, and one can recover Chevalley's formula for the action of $V \subseteq S$ on the cohomology ring in this basis. Later, for infinite crystallographic W, Kostant and Kumar used them to construct the cohomology ring of the flag variety of a Kac-Moody group and then the "dual" nil Hecke ring was shown to give the T-equivariant cohomology.

One can expand $x_w = \sum_v \delta_v A_{v,w}$ and $\delta_w = \sum_v \delta_v B_{v,w}$ for certain inverse matrices $(A_{v,w})$, $(B_{v,w})$ of rational functions. Remarkably given their definition, after trivial renormalization, they play a completely symmetric role despite the completely asymmetric role of δ_w and x_w above; the recurrence formulae for them differ only by replacing l by $-l$ (this turns out have a simple explanation in a general commutative algebra context).

The renormalized functions will be denoted $S_{x,w}$ below. There is a criterion [15] which describes the smooth points of Schubert varieties (for crystallographic groups) in terms of these functions. They also appear in certain multiplicity one criteria for representations in characteristic zero and p constructed using ideas in [12]. For dihedral groups, the numerators are binomial coefficient modeled on series of root coefficients (for the affine dihedral groups, the series of root coefficient is $0, 1, 2, 3, \dots$ and one recovers ordinary binomial coefficients.)

3.5. Sketch of a proof of Deodhar's conjecture. For $x \in W$, let $\sigma_x : R \to R$ denote the graded $\mathbb{R}$-algebra automorphism induced by $x^{-1} : V \to V$. The argument in [10] can be adapted to give an self-contained proof of Deodhar's conjecture using steps as follows.

(1) Show the graded free R-mod $M = M^{(l)}$ with basis m_x (of degree $2l(x)$) for $x \in W$, has a left R-module action making it an $R' := R \otimes_{\mathbb{R}} R$-modules given by

$$\chi m_x = m_x \sigma_x(\chi) - \sum_{\substack{x \xrightarrow{\alpha} y \\ l(y)=l(x)+1}} m_y \langle\, x, \alpha^\vee \,\rangle \tag{3.5.1}$$

for $\chi \in V = R_2$.

Proof. This is an algebraic version of the Chevalley formula for (T-equivariant) cohomology. One can check the definition gives a well defined action using the properties of labelings of length two intervals. Another proof can be obtained using following ideas: for $l = -l_0$, set $m_y = t_y$ where $t_y = \bar{x_y}$ from nil Hecke ring; for $l = l_0$, consider the dual basis of the (right) R-graded dual module $\mathrm{Hom}_R(M^{(-l_0)}, R)$. One then has (in either case)

$$t_r m_y = \begin{cases} m_{ry}, & ry < y \\ 0, & \text{otherwise.} \end{cases} \tag{3.5.2}$$

□

(2) Show that $M \otimes_R Q$ has a right Q-basis δ_x satisfying $\chi\delta_x = \delta_x \sigma_x(\chi)$ for all $\chi \in V$ such that $m_x = \sum_{y \geq x} \delta_y S^{(l)}_{x,y}$ where $S^{(l)}_{x,y} \in Q$, $S^{(l)}_{x,y} = 0$ unless $x \leq y$, $S^{(l)}_{x,x} = 1$.

Proof. For reverse Bruhat order and Bruhat order, this can be deduced from the construction of M in (1) (to obtain $S_{x,x} = 1$, one has to rescale the δ_x there by multiplication by certain elements of Q). Another proof uses a simple fact from homological algebra in [12] to show that the bimodule extensions between obvious subquotient bimodules of M (isomorphic as right $R \otimes R$-module to $\delta_x R$) split over Q. □

(3) Obtain recurrence formula

$$S^{(l)}_{x,y}(\sigma_x-\sigma_y)(\chi)=\sum_{\substack{x\xrightarrow{\alpha}z\\ l(z)=l(x)+1}}\langle\,\chi,\alpha^\vee\,\rangle S^{(l)}_{z,y} \tag{3.5.3}$$

for $\chi\in V$.

Proof. This follows directly by putting (2) in the Chevalley formula. □

(4) Obtain

$$0\neq S^{(l)}_{x,y}=\frac{\epsilon_l^{l(y)-l(x)}f^{(l)}_{x,y}}{\prod_{x^{-1}\leq z^{-1}\xrightarrow{\beta}y^{-1}}\beta} \tag{3.5.4}$$

which is homogeneous of degree $-2(l(y)-l(x))$, (i.e. numerator, denominator are homogeneous of degree n, d with $d-n=2(l(y)-l(x))$) and where $\epsilon_l=1$ if $l=l_0$ and $\epsilon_l=-1$ if $l=-l_0$.

Proof. For this, one takes $\chi=x(\chi')$ for some χ' in the interior of the fundamental chamber for V i.e. with $\langle\,\chi',\alpha\,\rangle>0$ for all $\alpha\in\Pi$. This implies that $\chi'-z(\chi')\in\mathbb{R}_{\geq 0}\Pi$ for all $z\in W$. The only linear factor of the denominator of $S^{(l)}_{x,y}$ not already present in some $S_{z,y}$ with $x<z\leq y$ is a constant times $(\sigma_x-\sigma_y)(\chi)=(1-y^{-1}x)(\chi')$; but if x,y are not joined by an edge of the Bruhat graph, the image of $(1-y^{-1}x)(V)$ contains two relatively prime (i.e. non-proportional) linear terms and neither can be in the denominator of $S^{(l)}_{x,y}$. □

(5) Since $n\geq 0$, get $d\geq 2(l(y)-l(x))$ and $|\{\,\beta\in\Phi_+\mid x^{-1}\leq z^{-1}\xrightarrow{\beta}y^{-1}\,\}|\geq l(y)-l(x)$. Since $\leq_l$ can be either Bruhat order or reverse Bruhat order, Deodhar's conjecture follows.

3.6. The "possibly infinite upper triangular" matrices $S^{(l_0)}_{x,y}$, $S^{(-l_0)}_{y,x}$, with $x,y\in W$, are mutually inverse:

$$\sum_y S^{(l_0)}_{x,y}S^{(-l_0)}_{z,y}=\delta_{x,z}=\sum_y S^{(l_0)}_{y,x}S^{(-l_0)}_{y,z} \tag{3.6.1}$$

Proof. This comes from an identification of $M^{(-l_0)}$ as the graded dual of $M^{(l_0)}$. □

3.7. In the argument of 3.5, noting that $0\neq\chi'-y^{-1}x(\chi')\in\mathbb{R}_{\geq 0}\Pi$, it follows that $f^{(l)}_{x,y}$ times some (nonzero) polynomial with nonnegative coefficients in the simple roots is itself a (nonzero) polynomial with non-negative coefficients in the simple roots.

Conjecture. Let $N:=l(y)-l(x)$. Above, $\epsilon_l^{l(y)-l(x)}S^{(l)}_{x,y}$ is expressible as a sum of terms each of the form $\frac{c}{\beta_1\ldots\beta_N}$ for certain scalars $c\geq 0$ and certain distinct $\beta_1\ldots,\beta_N\in\{\,\beta\in\Phi_+\mid x^{-1}\leq z^{-1}\xrightarrow{\beta}y^{-1}\,\}$.

Exercise. Compute $S_{srts,s}$ for the universal group (in reverse Bruhat order) and check the conjecture.

4. Volume formula for polyhedral cones

This section repeats the proof of Deodhar's conjecture mutatis mutandis for vector-labeled face lattice F of a polyhedral cone C (though here there is no good analogue of the nil Hecke ring, but only of the rational functions $S_{x,w}$ which determine its structure constants).

4.1. Let $l(x) := \dim(x)$ for $x \in F$. Let $R := \mathrm{Sym}(V)$ denote the symmetric algebra of V, $\mathbb{N}$-graded so $R_0 = \mathbb{R}$ and $R_2 = V$. Again let Q be the quotient field of R. For $x \in F$, let $\sigma_x \colon R \to R$ denote the graded $\mathbb{R}$-algebra homomorphism induced by the linear map $V \to V$ which is the orthogonal projection on $x^\perp$.

(1) Show the graded free R-mod $M = M^{(C)}$ with basis m_x (of degree $2l(x)$) for $x \in F$, has a left R-module action making it an $R' := R \otimes_{\mathbb{R}} R$-module with action given by

$$\chi m_x = m_x \sigma_x(\chi) - \sum_{x \xrightarrow{\alpha} y} m_y \langle x, \alpha \rangle \tag{4.1.1}$$

for $\chi \in V = R_2$.

Proof. Check using properties of the labeling. □

(2) Let Q be the quotient field of R. Show that $M \otimes_R Q$ has a right Q-basis δ_x satisfying $\chi\delta_x = \delta_x\sigma_x(\chi)$ for all $\chi \in V$ such that $m_x = \sum_{y \geq x} \delta_y S^{(C)}_{x,y}$ where $S^{(C)}_{x,y} \in Q$ and $S^{(C)}_{x,y} = 0$ unless $x \leq y$, $S_{x,x} = 1$.

Proof. By the homological algebra argument. □

(3) Obtain the recurrence formula

$$S^{(C)}_{x,y}(\sigma_x - \sigma_y)(\chi) = \sum_{x \xrightarrow{\alpha} z} \langle \chi, \alpha \rangle S^{(C)}_{z,y} \tag{4.1.2}$$

for $\chi \in V$.

Proof. As before. □

(4) Obtain

$$0 \neq S^{(C)}_{x,y} = \frac{f^{(C)}_{x,y}}{\prod_{x \leq z \xrightarrow{\beta} y} \beta} \tag{4.1.3}$$

which is homogeneous of degree $-2(l(y) - l(x))$.

Proof. As before. □

Remarks. The construction of $M^{(C)}$ is a special case of a family of modules for certain shellable subspace arrangements in vector spaces over general fields.applied here to the arrangement consisting of linear spans of the faces of the cone. (In general, there is one module for each element of the poset and in each module, the number of basis elements associated to an element of the poset, if nonzero, is the dimension of the top-dimensional cohomology of a corresponding open subinterval of the poset. In this case, the modules all all arise as submodules of a single module, and there is just 1 basis element for each element of the poset because the open intervals are spherical).

4.2. Identifying the face lattice of $C^\vee$ with F^{op} (which is F as a set) the matrices $S^{(C)}_{x,y}$, $S^{(C^\vee)}_{y,x}$, with $x, y \in F$ satisfy:

$$\sum_y (-1)^{l(y)-l(x)} S^{(C)}_{x,y} S^{(C^\vee)}_{z,y} = \delta_{x,z} = \sum_y (-1)^{l(y)-l(x)} S^{(C)}_{y,x} S^{(C^\vee)}_{y,z} \tag{4.2.1}$$

4.3. For $x \le y \in F$, recall that $C^{\uparrow}_{x,y}$ is the polyhedral cone $\mathbb{R}_{\ge 0}\{\,\alpha \mid x \xrightarrow{\alpha} z \le y\,\}$ and $C^{\downarrow}_{x,y}$ is the polyhedral cone $\mathbb{R}_{\ge 0}\{\,\alpha \mid x \le z \xrightarrow{\alpha} y\,\}$. These are mutually dual polyhedral cones (in their linear span) with face lattices naturally identified with $[x, y]$ and $[x, y]^{\mathrm{op}}$ respectively.

Let V^* be the dual space of V identified with V via $\langle -, - \rangle$. We identify $R = \mathrm{Sym}(V) = \mathrm{Sym}(V^*)$ with the algebra of real polynomial functions on V and Q with the algebra of rational functions on V.

Let $\mathrm{rVol}(P)$ (relative volume) denote the $\dim(P)$-volume of a convex set P in its affine span $\mathrm{aff}(P)$, with respect to measure induced by $\langle -, - \rangle$ on the translation space of $\mathrm{aff}(P)$.

Theorem. *Let $x \le y$ in F and $N := l(y) - l(x)$. If $\chi \in V$ with $\langle \chi, \alpha \rangle > 0$ for all α with $x \le z \xrightarrow{\alpha} y$, then $S^{(C)}_{x,y}(\chi) = (l(y) - l(x))! \cdot \mathrm{rVol}(\{\, v \in C^{\downarrow}_{x,y} \mid \langle v, \chi \rangle \le 1\,\}$.*

Proof. If P is a (positive-dimensional) convex polytope and $p \in \mathrm{aff}(P)$, then one has

$$\mathrm{rVol}(P) = \frac{1}{\dim(P)} \sum_F \langle\, u_F, p_F - p\,\rangle\, \mathrm{rVol}(F) \tag{4.3.1}$$

where F runs over the distinct facets (codimension-1 faces) of P, p_F is a point of $\mathrm{aff}(F)$ and u_F is the outward unit normal to P on F. (If p is in the relative interior of P, this follows by decomposing P into the union of pyramids with vertex p over the facets F. The independence of $\mathrm{rVol}(P)$ from the choice of such p then implies that this independence holds for all p.) The theorem follows by induction on N by comparison of (4.1.2) and (4.3.1), using the formula for relative volume of the pyramid $\{\, v \in C^{\downarrow}_{z,y} \mid \langle v, \chi \rangle \le 1\,\}$ (for $z = x$ or $x \lessdot z \le y$) in terms of the relative volume of its base $\{\, v \in C^{\downarrow}_{z,y} \mid \langle v, \chi \rangle = 1\,\}$ and its perpendicular height. ☐

Exercise. (a) Check the theorem for a rank two cone.
(b) Check the omitted details of the proof.

4.4. It follows from the preceding theorem that for any salient polyhedral cone C in V (i.e. $C \cap -C = \{0\}$) there is a rational function $\mathrm{vol}_C \in Q$, called the volume function of C, with the following property: for all $\chi \in V$ such that $\langle \chi, \alpha \rangle > 0$ for each $\alpha \in V$ such that $\mathbb{R}_{\geq 0}\alpha$ is an extreme ray of C, one has $\mathrm{vol}_C(\chi) = \dim(C)! \cdot \mathrm{rVol}(\{ v \in C \mid \langle v, \chi \rangle \leq 1 \})$. Note that the set of $\chi \in V$ with the above property is a non-empty open subset of V (in standard topology) and so vol_C is uniquely determined as an element of Q. Moreover, vol_C is homogeneous of degree $-2\dim(C)$ and is expressible with denominator $\prod \alpha$ where the prod is over a set of representatives of extreme rays of C (this gives a proof along the same lines as that of Deodhar's conjecture that a d-dimensional polyhedral cone has at least d extreme rays). The following shows the analogue of conjecture 3.7 holds for cones.

Corollary. *Let assumptions be as in the preceding theorem. Then $S^{(C)}_{x,y}$ is expressible as a sum of terms each of the form $\frac{c}{\beta_1 \ldots \beta_N}$ for certain scalars $c > 0$ and certain distinct $\beta_1, \ldots, \beta_N \in \{ \beta \in V \mid x \leq z \xrightarrow{\beta} y \}$ where $N = l(y) - l(x)$.*

Proof. Choose a polytopal base B for $C^{\downarrow}_{x,y}$, so every non-zero element of $C^{\downarrow}_{x,y}$ is uniquely expressible in the form λb with $\lambda \in \mathbb{R}_{>0}$ and $b \in B$. It is known that B has a triangulation in which the simplexes occurring have all their vertices amongst the vertices of B (see [16] or [1]). Taking cones over these simplexes gives rise to a corresponding decomposition of $C^{\downarrow}_{x,y}$ into simplicial cones C_i the extreme rays of which are amongst the extreme rays of $C^{\downarrow}_{x,y}$. Suppose that $C_1, \ldots, C_m$ are the N-dimensional simplicial cones (others have smaller dimension). Then for χ as in the theorem,

$$\mathrm{rVol}(\{ v \in C^{\downarrow}_{x,y} \mid \langle v, \chi \rangle \leq 1 \}) = \sum_{i=1}^{m} \mathrm{rVol}(\{ v \in C_i \mid \langle v, \chi \rangle \leq 1 \}). \tag{4.4.1}$$

This implies that $S_{x,y} = \mathrm{vol}(C^{\downarrow}_{x,y}) = \sum_{i=1}^{m} \mathrm{vol}(C_i)$. Now if $\beta_1, \ldots, \beta_N$ are representatives of the extreme rays of C_i, then for degree reasons, $\mathrm{vol}(C_i) = \frac{c}{\beta_1 \cdots \beta_N}$ for some $c \in \mathbb{R}$ which clearly must be positive. The corollary follows. □

4.5. Some natural questions on the volume functions vol_C follow.

Question. (a) Is there a natural interpretation of $\mathrm{vol}_C(\chi)$ for other $\chi \in V$ for which the rational function vol_C is defined? (for instance, as a signed volume of another polyhedral cone).
(b) For which polyhedral cones C, D is $\mathrm{vol}_C = \pm \mathrm{vol}_D$.
(c) What is the nature of the analogous function vol_C (where defined) for C a general closed salient cone?

4.6. Define the *cone group* $\mathscr{C}$ as the abelian group with generators $[C]$ for C a salient polyhedral cone in V subject to relations

$$[C_1 \cup C_2] = [C_1] + [C_2] - [C_1 \cap C_2] \tag{4.6.1}$$

whenever C_1, C_2 are polyhedral cones such that $C_1 \cup C_2$ is a salient polyhedral cone (note that $C_1 \cap C_2$ is then a salient polyhedral cone too). A valuation of $\mathscr{C}$ in an abelian group A is a group homomorphism $\nu\colon \mathscr{C} \to A$. For example, for $v \in V$, there is a valuation $\nu_v\colon \mathscr{C} \to \mathbb{Z}$ determined by $\nu_v([C]) = 1$ if $v \in C$ and $\nu_v([C]) = 0$ if $v \notin C$. Another valuation is determined by $[C] \mapsto 0$ if $\mathbb{R}C \neq V$ and $[C] \mapsto \mathrm{vol}_C$ if $\mathbb{R}C = V$. To make the relationship of the next conjecture to 4.2.1 clearer, we shall use the following notation: for salient polyhedral cones C, D with $\mathbb{R}C \cap \mathbb{R}D = 0$, write $[C] \cdot [D] = [C + D]$ (note $C + D$ is a salient polyhedral cone).

Conjecture. For $x < z$ in F, one has in $\mathscr{C}$

(a) $\sum_{z\in[x,y]}(-1)^{l(z)-l(x)}[C^{\downarrow}_{x,z}] \cdot [C^{\uparrow}_{z,y}] = 0.$
(b) $\sum_{z\in[x,y]}(-1)^{l(z)-l(x)}[C^{\uparrow}_{x,z}] \cdot [C^{\downarrow}_{z,y}] = 0.$

Informally, this claims that the two "matrices" with entries given by $[C^{\downarrow}_{x,y}]$ and $(-1)^{l(y)-l(x)}[C^{\uparrow}_{x,y}]$ for $x, y \in F$ are "mutually inverse."

Exercise. Check the conjecture for two-dimensional cones.

5. Conjecture on Bruhat intervals and polyhedral cones

5.1. The following conjecture asserts loosely that the undirected root-labelled Bruhat graph of a Bruhat interval arises as a degeneration of a vector-labelled Hasse diagram of some polyhedral cone in such a way that the Bruhat interval identifies with the face poset of a regular "polytopal" CW-complex (determined by the degeneration) whose cells are unions of faces of a base of the polyhedral cone.

Conjecture. Let Ω be a closed interval in W in $\leq_l$. Then there is a non-canonically associated vector space U, polyhedral cone C in U, a linear map $L\colon U \to V$, a labeling of covering edges of the Hasse diagram of the face lattice F of C, written $x \xrightarrow{\alpha} y$ for $x \lessdot y$ in F and a function $\iota\colon F \to \Omega$, with the following properties:

(a) The labeling of F has the properties in Proposition 2.4
(b) For an edge $x \xrightarrow{\alpha} y$ of F, have $\iota(x) \xrightarrow{uL(\alpha)} \iota(x)$ or $\iota(y) \xrightarrow{uL(\alpha)} \iota(x)$ where $u \in \mathbb{R}^*$ (units of $\mathbb{R}$) i.e. $\iota(x)$ and $\iota(y)$ are joined in Ω by an undirected edge with label $uL(\alpha) \in \Phi_+$.
(c) For $y \in [x, w]$, let $C_y := \cup z$ where the union is over all $z \in F$ such that $z \leq z'$ for some $z' \in F$ with $\iota(z') \leq y$. Fix any polytopal base B of C (the choice of B is immaterial) and set $\sigma_y = C_y \cap B$. Then there is a regular CW-complex underlying B with face poset Ω (including $\emptyset$) and underlying closed cells σ_y for $y \in \Omega$.

Example. Type A_2 can be done by a type of combinatorial Bott-Samelson resolution which works more generally.

5.2. Further parts of the conjecture concern compatibly with much natural additional structure attached to Bruhat intervals and polyhedral cones. Only a small portion of this structure has been described here (e.g. the rational functions $S_{x,w}$). Additional evidence that the conjecture is natural is its compatibility with constructions involving the Z-property. It would be very interesting to identify general conditions on a poset Ω and U, C, L, F, i as above which make Ω "Bruhat like."

Examples of posets Ω and such data not apparently coming from Bruhat intervals or face lattices of cones can be defined using the Z-property to glue together face lattices of suitably related polyhedral cones. One conjectures the analogues of Kazhdan-Lusztig polynomials for such structures are well-defined and have similar properties as conjectured for general Coxeter groups (e.g. non-negative coefficients). One may hope that development of a suitable framework related to the conjecture may provide a natural (more functorial) context for study even of classical Kazhdan-Lusztig polynomials for general Coxeter groups.

5.3. In the case when $x \xrightarrow{\alpha} y$ in Ω implies $x \lessdot y$; then one may choose ι to be poset isomorphism. There is trivial (algebraic) degeneration, but no difficult (combinatorial) degeneration and the conjecture and its elaborations can be proved in this case.

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